sults of Mr. Galbraith's analyses. Having had occasion to analyze the water of the river Liffey above King's Bridge, in order to ascertain the quantity of alkalies contained in it, he found distinct evidence of the presence of potash, and none whatever of the presence of soda. And as this river takes its rise in the granite platform of the Wicklow hills, and might be said to contain the washings of that district, the presence of potash strongly confirms the opinion maintained by Mr. Galbraith, that the felspar of the Dublin and Wicklow hills was potash felspar.

Mr. Galbraith explained, that he had used the terms orthose and albite in the sense in which Sir R. Kane had used them, although he did not consider it quite exact, as his object was to confine himself exclusively to the consideration of the relative numerical quantities of potash and soda in the Wicklow felspars.

The Rev. Professor Graves continued the reading of his Paper on the principles which regulate the interchange of symbols in certain symbolic equations.

Let π and ρ be two distributive symbols of operation, which combine according to the law expressed by the equation,

$$\rho\pi = \pi\rho + a, \tag{1}$$

a being a constant, or at least a symbol of distributive operation commutative with both π and ρ.

In this fundamental equation, if we change π into ρ and ρ into $-\pi$, it becomes

$$-\pi\rho = -\rho\pi + a,$$

or

$$\rho\pi = \pi\rho + a, \tag{2}$$

the same as before. From this it follows, that in any symbolical equation,

$$\phi(\pi, \rho) = 0, \tag{3}$$

which has been directly deduced from the fundamental equa-

tion (1), without any further assumption as to the nature of the operations denoted by π and ρ, we may change π into ρ, and ρ into $-\pi$; so as at once to form the correlative equation,

$$\phi(\rho, -\pi) = 0; \qquad (4)$$

for this latter will be deducible from the primitive, in the form (2), by the same processes, whatever they are, which conduct us from (1) to (3).

The value of this principle must depend upon the extent of its application; and this will be found much wider than might at first sight be supposed. For a symbolical equation of the form (3), which is verified for any subject whatever operated on by its left-hand member, if it be not, in its existing state, identically true, must hold good in consequence of our being able to transform it into one that is identically true by means of the fundamental equation (1), which connects π and ρ. Thus we may regard all useful equations of the form (3) as deductions from the single primitive (1).

Of the general nature of the results which may be deduced from this one very simple equation, and that without the introduction of any fresh hypothesis as to the operation of π and ρ, the following example will give a sufficient idea.

Making $a = 1$, which does not much diminish the generality of our conclusions, we have

$$\begin{aligned} \rho\pi &= \pi\rho + 1, \\ \rho\pi^2 &= \pi\rho\pi + \pi, \\ &= \pi(\pi\rho + 1) + \pi, \\ &= \pi^2\rho + 2\pi. \end{aligned}$$

Again,

$$\begin{aligned} \rho\pi^3 &= \pi^2\rho\pi + 2\pi^2, \\ &= \pi^2(\pi\rho + 1) + 2\pi^2, \\ &= \pi^3\rho + 3\pi^2. \end{aligned}$$

And for n any positive integer we get,

$$\rho\pi^n = \pi^n\rho + n\pi^{n-1}.$$

Again, operating on the equation

$$\rho\pi = \pi\rho + 1,$$

with

$$\pi^{-1}(\ )\pi^{-1},$$

we get

$$\rho\pi^{-1} = \pi^{-1}\rho - \pi^{-2};$$

from which we deduce

$$\rho\pi^{-n} = \pi^{-n}\rho - n\pi^{-n-1}.$$

So that the equation

$$\rho\pi^{n} = \pi^{n}\rho + n\pi^{n-1},$$

holds good for any integer value of n.

From this again we infer that

$$\rho\ \psi\pi = \psi\pi\ \rho + \psi'\pi, \qquad (5)$$

where $\psi\pi$ represents any function of integral powers of π. And from (5), finally, we can ascend to the more general theorem,

$$\phi\rho\ \psi\pi = \psi\pi\ \phi\rho + \psi'\pi\ \phi'\rho + \frac{1}{1.2}\psi''\pi\ \phi''\rho + \&c. \qquad (6)$$

Changing π into ρ, and ρ into $-\pi$, in the last two equations, we obtain the correlative ones,

$$\pi\ \phi\rho = \phi\rho\ \pi - \phi'\rho, \qquad (7)$$

$$\psi\pi\ \phi\rho = \phi\rho\ \psi\pi - \phi'\rho\ \psi'\pi + \frac{1}{1.2}\phi''\rho\ \psi''\pi - \&c. \qquad (8)$$

In the theorems here given the reader will recognise an extension to the symbols π and ρ of the theorems respecting x and D, stated by Dr. Hargreave at the commencement of his remarkable paper on the Solution of Differential Equations, printed in the Transactions of the Royal Society for 1848. Having obtained the theorems,

$$\phi D\ \psi x = \psi x\ \phi D + \psi' x\ \phi' D + \frac{1}{1.2}\psi'' x\ \phi'' D + \&c., \qquad (9)$$

$$\psi x\ \phi D = \phi D\ \psi x - \phi' D\ \psi' x + \frac{1}{1.2}\phi'' D\ \psi'' x - \&c., \qquad (10)$$

separately, Dr. Hargreave observed that the latter might be deduced from the former by changing x into D, and D into $-x$; and on this observed fact he founded the conclusion that, in expressions capable of being reduced to the form (9) or (10), we are at liberty to effect the above-mentioned interchange of symbols.

The preceding investigation enables us to account for the fact just referred to, and to establish on what seems to be its real foundation the validity of the proposed method of deriving formulæ one from the other. If we take $f(x)$ any function of x, we shall have

$$D\ (xfx) = x'xf + fx,$$

or, detaching the subject fx from the operations effected on it, we find that

$$Dx = xD + 1 \qquad (11)$$

is a symbolical equation which holds good whatever subject be operated on by each of its terms. It is, in fact, the fundamental equation which defines the law according to which x and D combine. And as in this equation we may change x into D, and D into $-x$; we may do the same in (9), or in any other equation derived from it.

From this one equation (11) the principal symbolic formulæ of the Differential Calculus can be deduced; and we may, therefore, regard a great part of it as included in that single branch of the Calculus of Operations which refers to the properties of symbols connected by the fundamental equation with which this Paper commences.

But there are other changes of symbols which may be made in formulæ deduced from the equation,

$$\rho\pi = \pi\rho + 1.$$

Since ρ is distributive, we shall have

$$\rho\ f\rho = f\rho\ \ \rho,$$

and adding these equations together, we get

$$\rho\,(\pi + f\rho) = (\pi + f\rho)\,\rho + 1, \tag{12}$$

an equation still of the same *form* as (1). And, therefore, in any symbolical equation deduced from (1) merely in virtue of its form, we are at liberty to change π into $\pi + f\rho$. Similar reasoning will show that in symbolical formulæ obtained in the same way, we may change ρ into $\rho + f\pi$. As particular cases of this we may observe that in any symbolical equation involving x and D, we are at liberty to change x into $x + fD$, or D into $D + fx$.

Again, if we operate on (5) with $(\psi'\pi)^{-1}$, it becomes

$$(\psi'\pi)^{-1}\rho\,\psi\pi = \psi\pi\,(\psi'\pi)^{-1}\rho + 1,$$

inasmuch as any two functions of π are commutative. Now this again is an equation of the form

$$\rho\pi = \pi\rho + 1,$$

$(\psi'\pi)^{-1}\rho$ being put for ρ, and $\psi\pi$ for π.

It follows then that in any deduction from (1) we may change

$$\rho \text{ into } (\psi'\pi)^{-1}\rho,$$

and

$$\pi \text{ into } \psi\pi. \tag{13}$$

In like manner, if we operate on (7) with $(\phi'\rho)^{-1}$, it becomes

$$\phi\rho\,(\phi'\rho)^{-1}\pi = (\phi'\rho)^{-1}\pi\,\phi\rho + 1\,;$$

showing that in any deduction from (1) we may change

$$\rho \text{ into } \phi\rho,$$

and

$$\pi \text{ into } (\phi'\rho)^{-1}\pi. \tag{14}$$

Writing x for π, and D for ρ, we learn from (13) that it is legitimate in symbolical formulæ to change x into ψx, and D

into $(\psi' x)^{-1} D$. This, in fact, reproduces the known rule for the change of the independent variable.

From (14) we conclude that the change of

$$D \text{ into } \phi D,$$

and

$$x \text{ into } (\phi' D)^{-1} \pi,$$

is a legitimate one. The validity of this change has not, we believe, been noticed before. It is unnecessary to adduce any more particular instances of the general law of interchange of symbols which may be established, viz.: If from (1) we can deduce any equation of the form

$$\mathrm{P}\Pi = \Pi\mathrm{P} + 1;$$

when P and Π can be expressed in terms of ρ and π, then, in any symbolical equation derived from (1), we are at liberty to change ρ into P, and π into Π.

Some very important deductions may be made from the equation (1). As a particular case of formula (5), we have

$$\rho e^{\psi\pi} = e^{\psi\pi}\rho + e^{\psi\pi}\psi'\pi,$$

therefore,

$$\rho + \psi'\pi = e^{-\psi\pi}\rho e^{\psi\pi};$$

whence we conclude, that

$$f(\rho + \psi'\pi) = e^{-\psi\pi} f\rho\, e^{\psi\pi}. \qquad (15)$$

That is to say, the symbol,

$$e^{-\psi\pi}(\quad)\,e^{\psi\pi},$$

operating on any function of ρ will change it into the corresponding function of $\rho + \psi'\pi$.

Changing π into ρ, and ρ into $-\pi$, we should find

$$f(\pi + \psi'\rho) = e^{\psi\rho} f\pi\, e^{-\psi\rho}, \qquad (16)$$

which shows that the symbol,

$$e^{\psi\rho}(\quad)\,e^{-\psi\rho},$$

operating on any function of π will change it into the corresponding function of $\pi + \psi'\rho$.

The substitution of x for π, and D for ρ, in (16), leads to a result which is of considerable value, viz., that

$$e^{\psi D} fx\, e^{-\psi D} = f(x + \psi' D).$$

If in this symbolical equation we suppose the subject to be unity, we shall have

$$e^{\psi D} fx\, e^{-\psi D} 1 = f(x + \psi' D)\, 1. \qquad (17)$$

This is a remarkable extension of Taylor's theorem, when stated in the symbolical form; and will be found useful in the interpretation of symbolical expressions which are met with in the solution of differential equations. In the development of the right-hand member of formulæ (15) and (16), the terms involving D may be all brought by means of the theorems (6) and (8) to the right or left hand at pleasure. The formulæ thus obtained will be found of considerable use.

In the deduction and statement of theorems involving π and ρ, we shall find it convenient to employ the symbols

$$\frac{d}{d\pi} \text{ and } \frac{d}{d\rho},$$

either of which denotes the operation of taking the *derivèe*, in an algebraic sense, of any function of the symbol involved in it. According to this definition $\frac{d}{d\pi}$ must operate on π only where it appears *explicitly;* and so for $\frac{d}{d\rho}$.

Hence $\frac{d}{d\pi}$ is inoperative on ρ, or any function of it, and is commutative with ρ. So also $\frac{d}{d\rho}$ is commutative with π, or any function of it. The two symbols $\frac{d}{d\pi}$ and $\frac{d}{d\rho}$ are plainly commutative with one another; but they combine respectively

with π and ρ, in conformity with the law expressed by the equations,

$$\frac{d}{d\pi}\pi = \pi\frac{d}{d\pi} + 1,$$

$$\frac{d}{d\rho}\rho = \rho\frac{d}{d\rho} + 1.$$

The formulæ (6) and (8) may now be expressed in the symbolical form,

$$\phi\rho \;\; \psi\pi = e^{\frac{d}{d\pi}\frac{d}{d\rho}}\psi\pi \;\; \phi\rho, \tag{18}$$

$$\psi\pi \;\; \phi\rho = e^{-\frac{d}{d\pi}\frac{d}{d\rho}}\phi\rho \;\; \psi\pi. \tag{19}$$

But it must be observed that they are no longer as general as they were in their original form. The equations (6) and (8) would hold good, whatever expression involving π and ρ was written to the right of each of their terms. Whilst the operation of the exponentials in (18) and (19) must be restricted to the terms $\psi\pi\ \phi\rho$ and $\phi\rho\ \psi\pi$, which immediately follow them, and not allowed to affect the subject operated on by these terms.

As $\frac{d}{d\rho}$ is commutative with π, and $\frac{d}{d\pi}$ with ρ, we may write the formulæ (18) and (19) in the form,

$$\phi\rho \;\; \psi\pi = \psi\left(\pi + \frac{d}{d\rho}\right)\phi\rho,$$

$$\psi\pi \;\; \phi\rho = \phi\left(\rho - \frac{d}{d\pi}\right)\psi\pi,$$

and, for the same reason, are at liberty to develope

$$\psi\left(\pi + \frac{d}{d\rho}\right)$$

according to ascending powers of either π or $\frac{d}{d\rho}$. A similar observation applies to the development of

$$\phi\left(\rho - \frac{d}{d\pi}\right).$$

The very general theorems already stated may be extended to any number of systems of variables connected by equations, such as define the mutual action of π and ρ. Thus, if

$$\rho\pi = \pi\rho + 1,$$

and

$$\rho_1\pi_1 = \pi_1\rho_1 + 1,$$

the symbols being otherwise mutually commutative, we shall have

$$f(\rho, \rho_1)\,\phi\,(\pi, \pi_1) = e^{\frac{d}{d\pi}\frac{d}{d\rho} + \frac{d}{d\pi_1}\frac{d}{d\rho_1}}\,\phi\,(\pi, \pi_1) f(\rho, \rho_1),$$

and so on for any number of pairs of symbols.

Again, as a generalization of the formula (15), we shall find, if ψ denotes a function of π and π_1,

$$f\left(\rho + \frac{d\psi}{d\pi}, \rho_1 + \frac{d\psi}{d\pi_1}\right) = e^{-\psi} f(\rho, \rho_1)\, e^{\psi}.$$

And, analogous to (16),

$$f\left(\pi + \frac{d\psi}{d\rho}, \pi_1 + \frac{d\psi}{d\rho_1}\right) = e^{\psi} f(\pi, \pi_1)\, e^{-\psi};$$

ψ denoting in this case a function of ρ and ρ_1. Writing x and y for π and π_1, and $\frac{d}{dx}$ and $\frac{d}{dy}$ for ρ and ρ_1 in these latter formulæ, we obtain results of considerable importance, the statement and discussion of which is reserved for the concluding part of this Paper.

The Secretary read a paper by W. H. Harvey, M.D., on the Marine Botany of Western Australia.

Robert Ball, LL.D., drew the attention of the Academy to the fact, that in the celebrated statue, known as the Dying